Aging and self-organized criticality in driven dissipative systems

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We study the noisy dynamics of a close relative to the original sandpile model. Depending on the type of noise and the time scale of observation, we find stationary fluctuations (similar to self-organized criticality) or an aging dynamics with punctuated equilibria, a decreasing rate of events and reset properties qualitatively similar to those of glassy systems, evolution models, and vibrated granular media.

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I. INTRODUCTION

The "pulse-duration memory effect" observed [1] in sliding charge density wave systems was explained by Coppersmith and Littlewood [2] using a *microscopic* nonlinear model of interacting degrees of freedom with a huge number of dynamically inequivalent attractors. Related work by Tang, Wiesenfeld, Bak, Coppersmith, and Littlewood [3] (henceforth TWBCL), emphasized that the relatively rare *minimally stable* attractors of this model are nonetheless those preferably selected by the dynamics. The sandpile model and the idea of self-organized criticality (SOC) then evolved [4] from the analysis of the TWBCL model, with its *poised state* being conceptually similar to a minimally stable state. A sand pile [5] reacts to small disturbances by releasing avalanches with a broad distribution of sizes, returning then to its poised state described by the angle of repose.

While SOC deals with the *stationary* fluctuations of extended systems, a wide class of systems is manifestedly nonstationary, since the relevant *macroscopic* variables slowly change in time at a *decelerating* rate. This implies a dependence of the data on the initial time and hence on the *age* of the system. Relevant examples are spin glasses [6–8], the evolution of bacterial cultures [9], evolution in rugged fitness landscapes [10–12], macro-evolution [13,14], granular systems [15,16], and Lennard-Jones glasses [17]. In spin glasses and glasses, aging behavior is usually analyzed in terms of functions with two time arguments as, e.g., magnetic correlations and linear response. Since at "short" times $t<t_w$ these fulfill the fluctuation dissipation theorem (FDT), one can infer that the system performs equilibrium-like fluctuations in this regime [18]. For $t>t_w$ the FDT is broken and the nonstationary nature of the dynamics becomes apparent. Intimately linked to nonstationarity is the reset capability of aging systems, i.e., the possibility of enhancing the rate of relaxation, thus "resetting" the system's apparent age to a smaller value by tweaking parameters such as, e.g., temperature and/or magnetic field [6–8].

Below we use the TWBCL model, whose attractors are explicitly known, for a case study of the aging of nonthermal systems with multiple metastability. Being particularly interested in the connection between the coarse grained dynamics and the attractor structure, we find it convenient to consider the presence of two dynamical regimes, pseudostationary for $t<t_w$ and nonstationary for $t>t_w$, together with the reset capability as the central properties of aging dynamics. These properties are shared by spin glasses and glasses, but not by, e.g., the Bak-Sneppen [19] evolution model, whose macroscopic variable (the average fitness) remains constant in time. Nonetheless, this model has other interesting age dependent properties, as discussed in Ref. [20].

II. THE TWBCL MODEL

In spite of its out most simplicity, the TWBCL model with added noise has interesting aging properties: The relevant macroscopic average, here called the degree of phase organization $\|x\|$, remains nearly constant on scales $t<t_w$, and the noise induced fluctuations are avalanches [SOC-like in two dimensions (2D)]. For $t>t_w$ a logarithmic decrease of $\|x\|$ becomes apparent, revealing that the attractors visited become more stable as the system ages. The decay of $\|x\|$ can be reset by a change of the elastic constant, whereby the system is rejuvenated. All this behavior can approximately be accounted for by a mechanism previously dubbed [21] *noise adaptation*, which is also present in the dynamics of populations evolving in the rugged landscape of the NK model [10].

Consider M "balls" arranged in an array (linear or square) and coupled to their neighbors via springs with elastic constant K. The balls are subject to friction, to a force with a sinusoidal spatial variation, and to a series of square pulses of amplitude E. In the limit of high damping, large field, and weak elastic coupling, the key dynamical features are captured by the simple update rule [2] reproduced below (with 1D notation):

$$z_j(t)=y_j(t)+K\Delta(y(t))_j+E+N_j(t),$$

$$y_j(t+1)=\mathrm{nint}(z_j(t)). \tag{1}$$

Here, t is the time in units of field cycles, z_j is the coordinate of the jth ball, Δ is the lattice Laplacian, nint(z) stands for the integer nearest to z, and N_j is the noise applied at site j.

For $N=0$, integer valued E and free or periodic boundary conditions, the attractor states of Eq. (1) satisfy [3] nint$(Kc)=0$, where $c=\Delta y$ is the curvature vector. The corresponding coordinates then fulfill

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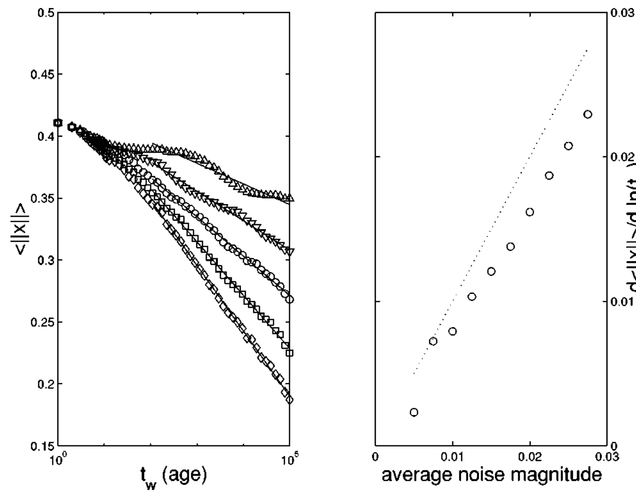

FIG. 1. Aging in a 1000×1 model with $K=0.05$, randomly perturbed by white noise with exponentially distributed magnitude of a. Each curve in the left panel belongs to a different value of a and is the average over ten independent trajectories, starting from the same minimally stable state. After a short transient, the decay is logarithmic (with a superimposed oscillation). In the right panel, the logarithmic slopes are shown vs a. A line of unit slope is included to guide the eye.

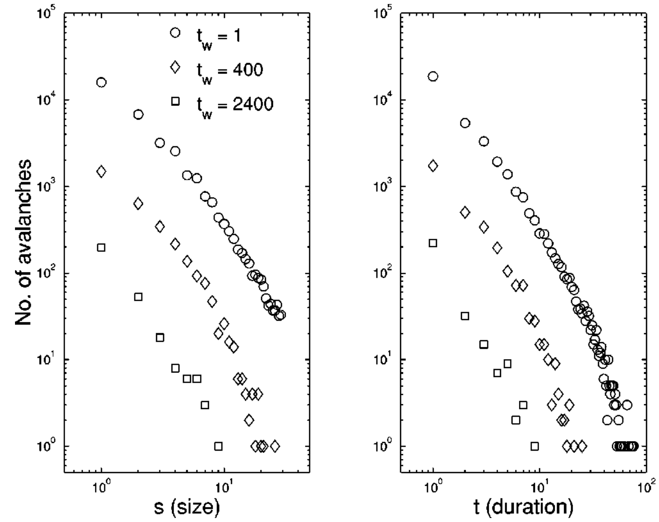

FIG. 2. Broad, power-law-like distributions are observed for both avalanche sizes (left panel) and durations in a 30×30 model with $K=0.05$. The system is subject to noise pulses drawn from an exponential distribution with average $a=0.01$ and the avalanches are monitored through 100 perturbation/relaxation cycles starting at three different ages t_w. The age is here the total number of cycles the system has undergone before sampling the statistics. As t_w increases the avalanches become smaller and shorter.

$$\frac{-1}{2}\leqslant Kc_j<\frac{1}{2}\quad j=1,\ldots,M. \tag{2}$$

The attractors [22] thus lie within an *attractor hypercube* of side length $1/K$ centered at the origin. Their number is $O(1/K^M)$, which is huge when, for example, $M\approx 200$ and $K=0.05$.

Noiseless relaxation of an initial state generally leads to a *phase organized state* [3], i.e., a state located at the corners of the attractor hypercube. Such state is *minimally stable* against external perturbations, as it barely fulfills Eq. (2). The average $\|x\|=M^{-1}\Sigma_i^M|x_i|$ is always defined and gauges, for attractors, the degree of (meta) stability $\|x\|$, or, equivalently, the *depth* $d=1/2-\|x\|$. Minimally stable attractors have $d\approx 0$.

Noisy relaxation properties

We always start the noisy dynamics at a phase organized state selected under noiseless conditions, and denote the time elapsed under the influence of noise by t_w, the age of the system. As we anticipated, the evolution has a first (pseudo) stationary phase involving fluctuations among metastable states of the same depth. On longer time scales the average depth of the attractors visited increases logarithmically through a series of jumps, also denoted *macroscopic events* or *punctuations*. Crucially, t_w demarks the boundary between short and long time dynamics. As shown by Fig. 1, our macroscopic average $\|x\|$ decreases in a logarithmic fashion, apart from a superimposed oscillation that is most clearly visible for small noise amplitudes. Let c be the logarithmic slope of $\|x\|$, which is shown in the second panel of Fig. 1 and assume that the observation window extends from t_w to $t+t_w$. Since $\ln(t+t_w)=\ln(t_w)+\ln(1+t/t_w)$, it follows that

$$\|x\|(t+t_w)\approx\|x\|(t_w)-c\ln(1+t/t_w). \tag{3}$$

Considering that $\ln(1+t/t_w)\approx t/t_w$ and that $c\ll 1$, we see that $\|x\|$ does not change appreciably as long as $t/t_w<1$. Hence, the dynamics appears stationary for $t<t_w$, as claimed.

We can reach the same conclusion by a second argument, which will help us to connect with the landscape structure of the problem: By definition, consecutive macroscopic events always delimit the observation window during which the dynamics appears as stationary. Second, as we will show later, the residence time t_r characterizing the attractors that are first visited at time t_w fulfills (within an order of magnitude)

$$t_r\approx t_w. \tag{4}$$

Hence the dynamics appears stationary within the interval $t<t_r\approx t_w$. Interestingly, Eq. (4) constitutes the main assumption of *weakly broken ergodicity* [24], a widely used scenario for complex system relaxation. The same equation also describes a property of diffusion on *hierarchical tree models* [23,25,26], models that reproduce many features of glassy relaxation.

Figure 2 illustrates the nature of the "short time" avalanche dynamics. The noise used to produce the data consists of a series of "kicks" of either sign, simultaneously applied to each "ball" and independently drawn from an exponential distribution with average a.

Relaxation to a fixed point is allowed between consecutive perturbations, and the avalanches consist of sets of contiguous "balls" simultaneously in motion. Their sizes are defined as the largest number of participating balls. Both size and duration are exponentially distributed in 1D, and power-law distributed in 2D, as expected [27]. The main message of

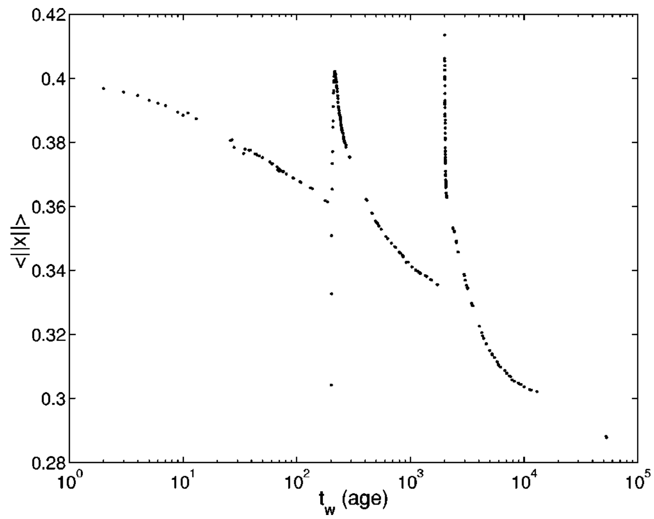

FIG. 3. Changing the elastic constant from $K=0.03$ to $K=0.05$ and again to $K=0.07$ produces the "resets" seen at times 2×10^3 and 2×10^4. The data are averages over 20 different trajectories of a linear array of 1000 "balls." The noise magnitude is exponentially distributed with average $a=0.015$.

Fig. 2 is simply that avalanches are larger and last longer in a young system (upper curve) than in an aged one (lower curve). This is a further indication of decelerating dynamics.

Pulsed noise can model systems where the typical avalanche duration and the length of the noiseless periods are well separated [28]. To bypass this restriction we now apply the noise "continuously," i.e., at each time step. The coarse grained time evolution in state space can then be followed by monitoring $\|x\|(t_w)$.

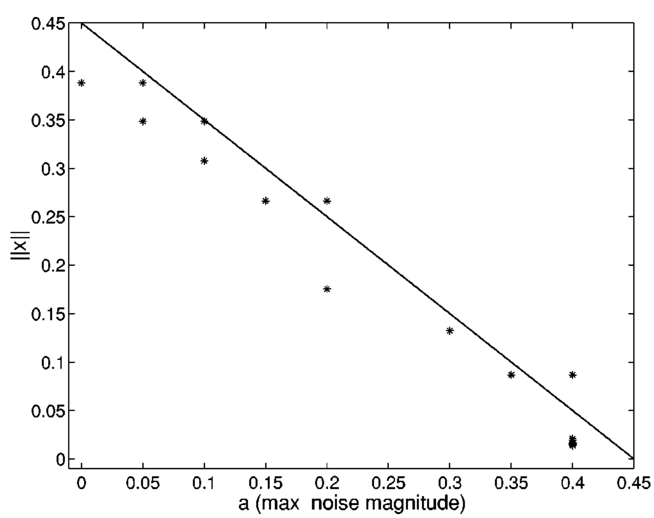

FIG. 4. A 1000×1 model with $K=0.05$ is perturbed through a few thousand updates by noise of bounded variation $[-a,a]$ and then allowed to fully relax, reaching the $\|x\|$ values that are plotted (as stars) vs a for five independent noise sequences. In addition, the line $y=0.45-a$ is drawn as a guide to the eye. Since $\|x\|$ decreases almost proportionally to a, the *least stable* among the attractors surviving the noise are those dynamically selected with high probability.

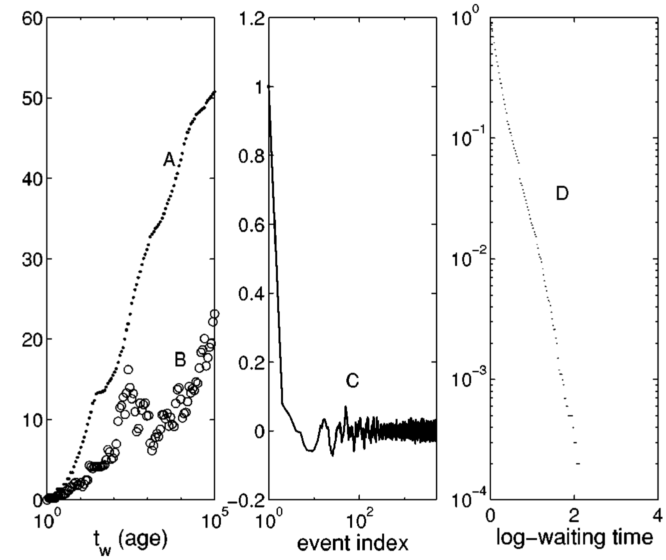

FIG. 5. An array of 1000 "balls" with elastic constant $K=0.05$ is perturbed by noise "kicks" of random sign and magnitude exponentially distributed with average $a=0.015$. The noise is uncorrelated in time and space. We considered 200 independent trajectories, all starting from the same minimally stable state. An "event" is defined as the achievement of a state of lower $\|x\|$. A and B: average and variance of the number of events observed within time t_w. C: autocorrelation function $C_\Delta(k)$ of the log-waiting times $\Delta_k=\ln(t_k/t_{k-1})$. D: distribution of the Δ_k.

Averaging suppresses all fast fluctuations together with the spatial information, and $\|x\|(t_w)$ therefore consists of constant plateaus, punctuated by rapid changes. These mainly lead to deeper attractors and stand out as *the* coarse grained dynamical events on long time scales.

Complementary information is obtained by averaging $\|x\|$ over independent noise histories. The resulting smooth function, $\langle\|x\|\rangle(t_w)$ was studied for both 1D and 2D systems, using pulsed as well as continuous noise. Since $\langle\|x\|\rangle$ is rather insensitive to the dimension, we mainly studied it for 1D models, which are faster to simulate. The left panel of Fig. 1 shows the time evolution of $\langle\|x\|\rangle(t_w)$ for different values of the noise magnitude a. The negative logarithmic slope of the plots has a linear relationship to the amplitude a that is shown in the right panel of the same figure.

The age reset is induced in our model by changing the elastic constant, which is analogous to changing the magnetic field [29] in spin glass systems. Increasing K reduces the size of the attractor hypercube and concomitantly reduces the depth of the current state. As a consequence $\langle\|x\|\rangle(t_w)$ is reset to an earlier (and larger) value, as shown in Fig. 3. Decreasing K has the effect of swelling the attractor hypercube whence $\langle\|x\|\rangle$ quickly drops.

To further clarify the connection between the reset effect and the attractor geometry, we consider *bounded noise* [21] drawn from a uniform distribution supported in the finite interval $[-a,a]$. Equation (1) then implies that only states fulfilling $\max_j\{1/2-|x_j|\}>a$ survive as *exact* fixed points of the equations of motion. The corner states of an attractor hypercube of side length $(1-2a)/K$ are still minimally stable, in the generalized sense that any infinitesimal *increase* of the noise amplitude destroys their stability. Figure 4 demonstrates that a perturbation of magnitude a permanently leaves the system in an attractor of depth $\approx 0.45 - a$. Hence the minimally stable states are dynamically selected.

For exponentially distributed noise of average magnitude a and on a time scale t_w, the kicks normally fall within the range $r_t \approx a \ln t_w$. Hence we expect that the trajectories will typically be located at the corners of a hypercube of side length $2\langle\|x\|\rangle \approx (1-2a \ln t_w)$. This is in reasonable agreement with the behavior depicted in the right panel of Fig. 1 and explains why even a modest shrinking of hypercube produces a sizable reset. Since the attractors typically discovered on a given time scale are the shallowest among those available, we expect that a *record* in the sequence of noise kicks will likely suffice to produce a macroscopic event, a feature previously dubbed *noise adaptation* [21].

If macroscopic events are induced by noise records, their number $n_e(t_w)$ during time t is a *log-Poisson* process [21]. As a consequence, if t_k denotes the time of the kth event, the quantities $\Delta_k = \ln(t_k/t_{k-1})$ are statistically independent and have the common distribution $\mathrm{Prob}(\Delta > x) = \exp(-\lambda x)$, for some positive λ. Second, the average number of events grows as $\langle n_e\rangle(t_w) = \lambda \ln t_w$. From Fig. 5 we see that the actual statistics of macroscopic events resembles a log-Poisson statistics in the shape of $\langle n_e\rangle$ (plot A) and in the fact that the log-waiting times have very short correlations (plot C) and an exponential distribution (plot D).

Crucially, Fig. 5 D shows that the event times t_k approximately make up a geometrical series. Hence, in a system of age t_w, most time is spent in the neighborhood of the last attractor visited. Therefore, the residence time in a neighborhood of the kth attractor discovered is $t_r = t_k - t_{k-1} \approx t_k = t_w$, as anticipated in Eq. (4).

III. SUMMARY AND CONCLUSION

On short time scales the space resolved dynamics of the TWBCL model can be described in terms of avalanches having, in two spatial dimensions, SOC-like character. On longer time scales the applied noise pushes the system into gradually more stable attractors. As a consequence, the degree of phase organization $\|x\|$ decreases logarithmically in time. This differs from the sand pile model and is reminiscent of the logarithmic relaxation of the angle of repose [15] observed in actual sand piles subject to vibration. We have argued that the aging of the TWBCL model is similar to that of, e.g., spin glasses in two important respects: (i) the boundary between quasistationary and nonstationary dynamics is given by t_w, the time elapsed from the initial quench, and (ii) the dynamics is resetable. The coarse grained aging dynamics is characterized by a series of ''macroscopic events'' leading to gradually deeper attractors. The statistics of these events is approximately log Poisson, an indication that the events themselves are strongly correlated with records in the history of noise. During noise adaptation [21] the attractors first visited on a time scale t_w typically trap the trajectories for time t_w, as assumed in weakly broken ergodicity [24]. The same statistics is also present in the dynamics of a population of ''agents'' evolving in NK fitness landscapes with multiple optima [10]. If one views evolution as a search in a fitness landscape with multiple optima, stress-induced hypermutation [30] following a change of nutrient type or concentration appears as the biological counterpart of a reset. Thus, a range of complex nonstationary phenomena can be qualitatively understood by invoking marginal stability and noise adaptation as selection mechanisms for metastable attractors.

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